The infinitesimal form of induced representation of the $\kappa-$Poincaré group

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Abstract

The infinitesimal form of the induced representation of the $\kappa-$Poincaré group is constructed. The infinitesimal action of the $\kappa-$Poincaré group on the $\kappa-$Minkowski space is described. The actions of these two infinitesimal forms on the solution of Klein-Gordon equation are compared.

1 Introduction

Recently, considerable interest has been paid to the deformations of group and algebras of space-time symmetries [7]. An interesting deformation of the Poincarè algebra [8], [6] as well as group [1] has been introduced which depend on the dimensional deformation parameter κ; the relevant objects are called $\kappa-$Poincarè algebra and $\kappa-$Poincarè group, respectively. Their structure was studied in some detail and many of their properties are now well understood. In particual, the induced representations of the deformed group were found [3] and the duality between $\kappa-$Poincarè group and $\kappa-$Poincarè algebra was also given [4]. Having the representations of the $\kappa-$Poincarè group and duality relations one can consider the infinitesimal form of the representation; in order to check whether we obtain the representation of the $\kappa-$Poincarè algebra. This is nontrivial as it is known from the construction of the induced representations the action of the Lorentz group on $q-$space is standard. Therefore the support spaces are described by the classical equation $q^2 = const$. On the other hand q^2 is not the Casimir operator of $\kappa-$Poincarè algebra.

In section 2 we describe the general definition of the induced representation of quantum group, than we find the infinitesimal form of the induced representation of $\kappa-$Poincaré group constructed by [3]. We show that in the massive case the infinitesimal form of the induced representation is the representation of the

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$\kappa-$Poincarè algebra. Next in section 3 we consider the infinitesimal action of $\mathcal{P}_\kappa$ on $\kappa-$Minkowski space $\mathcal{M}_\kappa$, and describe the Klein-Gordon equation. And on the end we compare actions of our two infinitesimal forms over the solution K-G equation.

In that paper we asume that $g_{\mu\nu}$ is diagonal $(+,-,-,-)$ tensor matric.

Let us remind the definition of $\kappa-$Poincarè group. The $\kappa-$Poincarè group $\mathcal{P}_\kappa$ is the Hopf *-algebra generated by selfadjoint elements $\Lambda^\mu{}_\nu,\ v^\mu$ subject to the following relations:

$$\begin{aligned}
[\Lambda^\alpha{}_\beta, v^\varrho] &= -\frac{i}{\kappa}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\varrho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\varrho}),\\
[v^\varrho, v^\sigma] &= \frac{i}{\kappa}(\delta^\varrho{}_0 v^\sigma - \delta^\sigma{}_0 v^\varrho),\\
[\Lambda^\alpha{}_\beta, \Lambda^\mu{}_\nu] &= 0.
\end{aligned}$$

The comultiplication, antipode and counit are defined as follows:

$$\begin{aligned}
\Delta\Lambda^\mu{}_\nu &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu,\\
\Delta v^\mu &= \Lambda^\mu{}_\nu \otimes v^\nu + v^\mu \otimes I,\\
S(\Lambda^\mu{}_\nu) &= \Lambda_\nu{}^\mu,\\
S(v^\mu) &= -\Lambda_\nu{}^\mu v^\nu,\\
\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu,\\
\varepsilon(v^\mu) &= 0.
\end{aligned}$$

Its dual structure, the $\kappa-$Poincarè algebra $\tilde{\mathcal{P}}_\kappa$ (in the Majid and Ruegg basis [9]) is a quantized universal envoloping algebra in the sense of Drinfeld [10] described by the following relations:

$$\begin{aligned}
[M_{ij}, P_0] &= 0,\\
[M_{ij}, P_k] &= i(g_{jk}P_i - g_{ik}P_j),\\
[M_{i0}, P_0] &= iP_i,\\
[M_{i0}, P_k] &= -i\frac{\kappa}{2}g_{ik}(1 - e^{\frac{-2}{\kappa}P_0}) + \frac{i}{2\kappa}g_{ik}P^rP_r - \frac{i}{\kappa}P_iP_k,\\
[P_\mu, P_\nu] &= 0,\\
[M_{ij}, M_{rs}] &= i(g_{is}M_{jr} - g_{js}M_{ir} + g_{jr}M_{is} - g_{ir}M_{js}),\\
[M_{i0}, M_{rs}] &= -i(g_{is}M_{r0} - g_{ir}M_{s0}),\\
[M_{i0}, M_{j0}] &= -iM_{ij}.
\end{aligned} \tag{1.1}$$

The coproducts, counit and antipode:

$$\begin{aligned}
\Delta P_0 &= I \otimes P_0 + P_0 \otimes I,\\
\Delta P_k &= P_k \otimes e^{-\frac{P_0}{\kappa}} + I \otimes P_k,
\end{aligned}$$

$$
\begin{aligned}
\Delta M_{ij} &= M_{ij} \otimes I + I \otimes M_{ij}, \\
\Delta M_{i0} &= I \otimes M_{i0} + M_{i0} \otimes e^{-\frac{P_0}{\kappa}} + \frac{1}{\kappa} M_{ij} \otimes P_j, \\
\varepsilon(M_{\mu\nu}) &= 0; \quad \varepsilon(P_\nu) = 0, \\
S(P_0) &= -P_0, \\
S(P_i) &= -e^{\frac{P_0}{\kappa}} P_i, \\
S(M_{ij}) &= -M_{ij}, \\
S(M_{i0}) &= -e^{\frac{P_0}{\kappa}} (M_{i0} - \frac{1}{\kappa} M_{ij} P_j),
\end{aligned}
$$

where $i, j, k, r, s = 1, 2, 3$.

Fact that the $\kappa-$Poincarè algebra is dual to the $\kappa-$Poincarè group was proved by [4] The fundamental duality relations read:

$$
\begin{aligned}
< P_\mu, f(v) > &= i \, \frac{\partial}{\partial v^\mu} f(v) \Big|_{v=0} \\
< M_{\mu\nu}, f(\Lambda) > &= i \, (\frac{\partial}{\partial \Lambda^{\mu\nu}} - \frac{\partial}{\partial \Lambda^{\nu\mu}}) f(\Lambda) \Big|_{\Lambda = I}
\end{aligned}
\tag{1.2}
$$

2 The infinitesimal form of the induced representation

Let us recall the definition of the representation of a quantum group $\mathcal{A}(G)$, acting in the linear space $\mathcal{V}$. It is simply a map

$$\varrho : \mathcal{V} \to \mathcal{V} \otimes \mathcal{A}(G)$$

satisfying

$$(I \otimes \Delta) \otimes \varrho = (\varrho \otimes I) \otimes \varrho$$

The induced representations are defined as follow [3], [11]: given any quantum group $\mathcal{A}(G)$, its quantum subgroup $\mathcal{A}(H)$ and the representation ϱ of the latter acting in the linear space $\mathcal{V}$, we consider the subspace of the space $\mathcal{V} \otimes \mathcal{A}(G)$ defined by the coequivariance condition:

$$\tilde{\mathcal{V}} = \{F \in \mathcal{V} \otimes \mathcal{A}(G) : id \otimes (\Pi \otimes id) \circ \Delta_G F = (\varrho \circ id) F\}$$

where $\Pi : \mathcal{A}(G) \to \mathcal{A}(H)$ is epimorfizm defining the subgroup $\mathcal{A}(H)$.

The induced representation is defined as a right action:

$$
\begin{aligned}
\tilde{\varrho} &: \tilde{\mathcal{V}} \to \tilde{\mathcal{V}} \otimes \mathcal{A}(G) \\
\tilde{\varrho} &= id \otimes \Delta
\end{aligned}
$$

In the paper [3] Maślanka obtained the following form of the induced representation in the massive case:

(elements of $\tilde{\mathcal{V}}$ we write $f(q_\mu) = e_i \otimes f_i(q_\mu)$, where $\{e_i\}$ are a basis of $\mathcal{V}$)

$$\begin{aligned} \varrho_{\mathcal{R}} : f_i(q_\nu) \quad \longmapsto \quad & \mathcal{D}_{ij}(\mathcal{R}(\widetilde{q}, \Lambda)) \cdot \\ & \exp(-i\kappa \ln(\cosh(\frac{m}{\kappa}) - \frac{q_0}{m}\sinh(\frac{m}{\kappa})) \otimes v^0) \cdot \\ & \exp(\frac{i\kappa \sinh(\frac{m}{\kappa}) q_k}{m\cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \otimes v^k) f_j(q_\mu \otimes \Lambda^\mu{}_\nu) \end{aligned} \tag{2.1}$$

where:

$$\begin{aligned} q_\mu &= m\Lambda^0{}_\mu \\ \widetilde{q}_0 &= \frac{mq_0 \cosh(\frac{m}{\kappa}) - m^2 \sinh(\frac{m}{\kappa})}{m\cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \\ \widetilde{q}_k &= \frac{mq_k}{m\cosh(\frac{m}{\kappa}) - q_0 \sinh(\frac{m}{\kappa})} \end{aligned}$$

and $f(q)$ are the square integrable functions defined on the hyperboloid $q^2 = m^2$, ($q_0 = \sqrt{q_i q_i + m^2}$) and taking values in the vector space carrying the unitary representation of the rotation group, the matrices $\mathcal{D}_{ij}$ are constructed in the same way as the matrices of the representation of classical orthogonal group and $\mathcal{R}(\widetilde{q}, \Lambda)$ is a classical Wigner rotation corresponding to the momentum $\widetilde{q}$ and transformation Λ, [3]. Of course, the right-hand side of eq.(2.1) is to be understood here as an element of the tensor product of the algebra of function on the hyperboloid $q^2 = m^2$ and the group $\mathcal{P}_\kappa$. Following Woronowicz [12] we define the infinitesimal form of the induced representation. For any element X of enveloping algebra $\tilde{\mathcal{P}}_\kappa$ and for any $f \in \mathcal{P}_\kappa$ if

$$X(f) = < X, f >$$

and

$$\varrho_{\mathcal{R}} : \mathcal{V} \to \mathcal{V} \otimes \mathcal{P}_\kappa$$

we define

$$\widetilde{X} : \mathcal{V} \to \mathcal{V}$$

by

$$\widetilde{X} = (I \otimes X) \circ \varrho_{\mathcal{R}}$$

Using the duality relations (1.2) after some calculi we arrive at the following formulas describing the infinitesimal form of our representation:

$$\begin{aligned} \widetilde{M}_{ij} &= i(q_i \frac{\partial}{\partial q^j} - q_j \frac{\partial}{\partial q^i}) + \varepsilon_{ijk} s_k \\ \widetilde{M}_{i0} &= -iq_0 \frac{\partial}{\partial q^i} + \varepsilon_{ijk} \frac{q_j s_k}{q_0 + m} \end{aligned}$$

$$
\begin{aligned}
\widetilde{P}_0 &= p_0 = \kappa \ln(\cosh(\frac{m}{\kappa}) - \frac{q_0}{m}\sinh(\frac{m}{\kappa})) \\
\widetilde{P}_j &= p_j = -\kappa \frac{\sinh(\frac{m}{\kappa})q_j}{m\cosh(\frac{m}{\kappa}) - q_0\sinh(\frac{m}{\kappa})}
\end{aligned} \tag{2.2}
$$

where $s_k,\ (k = 1, 2, 3)$ are the infinitesimal forms of representation $\mathcal{D}_{ij}$ (The representation of $SU(2)$ algebra).

It is easy to check that our operators satysfying relations of the $\kappa-$Poincaré algebra (1.1), so the infinitesimal form is the representation of the $\kappa-$Poincaré algebra.

3 The $\kappa-$Minkowski space, the infinitesimal action, K-G equation

The $\kappa-$Minkowski space ([1], [2]) $\mathcal{M}_\kappa$ is a universal $*$-algebra with unity generated by four selfadjoint elements x^μ subject to the following conditions:

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu).$$

Equipped with the standard coproduct:

$$\Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu,$$

antipode $S(x^\mu) = -x^\mu$ and counit $\varepsilon(x^\mu) = 0$ it becomes a quantum group.

The product of generators x^μ will be called normally ordered if all x^0 factors stand leftmost. This definition can be used to ascribe a unique element $: f(x) :$ of $\mathcal{M}_\kappa$ to any polynomial function of four variables f. Formally, it can be extended to any analytic function f.

Let us now define the infinitesimal action of $\mathcal{P}_\kappa$ on $\mathcal{M}_\kappa$. The $\kappa-$Minkowski space carries a left-covariant action of $\kappa-$Poincaré group $\mathcal{P}_\kappa$, $\varrho_L : \mathcal{M}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$, given by

$$\varrho_L(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu + v^\mu \otimes I. \tag{3.1}$$

Let X be any element of the Hopf algebra dual to $\mathcal{P}_\kappa$ – the $\kappa-$Poincaré algebra $\tilde{\mathcal{P}}_\kappa$. The corresponding infinitesimal action:

$$\widehat{X} : \mathcal{M}_\kappa \to \mathcal{M}_\kappa$$

is defined as follows: for any $f \in \mathcal{M}_\kappa$,

$$\widehat{X} f = (X \otimes I) \circ \varrho_L(f).$$

The following forms of generators were obtained in [2]:

$$\widehat{P}_\mu : f : \quad = \quad : i\frac{\partial f}{\partial x^\mu} :$$

$$
\begin{aligned}
\widehat{M}_{ij} : f : &= : -i(x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}) f : \\
\widehat{M}_{i0} : f : &= : \left[i x^0 \frac{\partial}{\partial x^i} - x_i \left(\frac{\kappa}{2}(1 - e^{-\frac{2i}{\kappa}\frac{\partial}{\partial x^0}}) - \frac{1}{2\kappa}\Delta \right) \right. \\
&+ \left. \frac{1}{\kappa} x^k \frac{\partial^2}{\partial x^k \partial x^i} \right] f :
\end{aligned}
$$

In that case these operators are not satysfying relation of $\kappa-$Poincaré algebra (1.1), becouse the action of $\kappa-$Poincarè algebra (group) on the $\kappa-$Minkowski space is antirepresentation (not representation). It is clear from the following equation:

$$
< ABf(P), \varphi(x) >=< f(P), \widehat{B}\widehat{A}\varphi(x) >
$$

for $A, B, P \in \tilde{\mathcal{P}}_\kappa$.

The deformed Klein-Gordon equation we write in the two equivalent forms:

$$
\left(\partial + \frac{m^2}{8} \right) f = 0,
$$

or

$$
\left[\partial_0^2 - \partial_i^2 + m^2 (1 + \frac{m^2}{4\kappa^2}) \right] f = 0,
$$

where operators $\partial_0,\ \partial_i,\ \partial$ are defined:

$$
\begin{aligned}
\partial_0 : f : &= : \left(\kappa \sin(\frac{1}{\kappa}\frac{\partial}{\partial x^0}) + \frac{i}{2\kappa} e^{\frac{i}{\kappa}\frac{\partial}{\partial x^0}} \Delta \right) f : \\
\partial_i : f : &= : (e^{\frac{i}{\kappa}\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^i}) f : \\
\partial : f : &= : \left(\frac{\kappa^2}{4}(1 - \cos(\frac{1}{\kappa}\partial_0)) - \frac{1}{8} e^{\frac{i}{\kappa}\partial_0} \Delta \right) f : \ .
\end{aligned}
$$

We can write the solution of K-G equation in the following wave function [5]:

$$
\Phi(x^\mu) = : \int \frac{d^3\vec{q}}{q^0} a(\vec{q}) e^{-ip_\mu(q) x^\mu} :
$$

where p_μ is deformed of q_μ defined in eq.(2.2).

For $\widehat{X} = \widehat{M}_{i0},\ \widehat{M}_{ij},\ \widehat{P}_\mu$, following the paper [5], let us define the operators $\mathcal{X}(q)$ by the following relation:

$$
\widehat{X}(q_\nu)\Phi(x^\mu) = : \int \frac{d^3\vec{q}}{q^0} \{ \mathcal{X}(q_\nu) a(\vec{q}) \} e^{-ip_\mu(q) x^\mu} :
$$

It is easy to see that:

$$
\mathcal{M}_{ij}(q) = -i(q_i \frac{\partial}{\partial q^j} - q_j \frac{\partial}{\partial q^i}) = -\widetilde{M}_{ij}(q)
$$

$$
\begin{aligned}
\mathcal{M}_{i0}(q) &= iq_0 \frac{\partial}{\partial q^i} = -\widetilde{M}_{i0}(q) \\
\mathcal{P}_{\mu}(q) &= p_{\mu}(q) = \widetilde{P}_{\mu}(q)
\end{aligned}
$$

The last relations compare generators defined in [5] from with our found from induced representation.

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